Exercise 3

Solve the differential equation or initial-value problem using the method of undetermined coefficients.

$$9y'' + y = e^{2x}$$

Solution

Since the ODE is linear, the general solution can be written as the sum of a complementary solution and a particular solution.

$$y = y_c + y_p$$

The complementary solution satisfies the associated homogeneous equation.

$$9y_c'' + y_c = 0 \tag{1}$$

This is a linear homogeneous ODE, so its solutions are of the form $y_c = e^{rx}$.

$$y_c = e^{rx} \quad \rightarrow \quad y_c' = re^{rx} \quad \rightarrow \quad y_c'' = r^2 e^{rx}$$

Plug these formulas into equation (1).

$$9(r^2 e^{rx}) + e^{rx} = 0$$

Divide both sides by e^{rx}.

$$9r^2 + 1 = 0$$

Solve for r.

$$r^2 = -\frac{1}{9}$$

$$r = \left\{-\frac{i}{3}, \frac{i}{3}\right\}$$

Two solutions to the ODE are $e^{-ix/3}$ and $e^{ix/3}$; by the principle of superposition, then,

$$\begin{aligned} y_c(x) &= C_1 e^{-ix/3} + C_2 e^{ix/3} \\ &= C_1\left(\cos\frac{x}{3} - i\sin\frac{x}{3}\right) + C_2\left(\cos\frac{x}{3} + i\sin\frac{x}{3}\right) \\ &= (C_1 + C_2)\cos\frac{x}{3} + (-iC_1 + iC_2)\sin\frac{x}{3} \\ &= C_3\cos\frac{x}{3} + C_4\sin\frac{x}{3}. \end{aligned}$$

On the other hand, the particular solution satisfies the original ODE.

$$9y_p'' + y_p = e^{2x} \tag{2}$$

Since the inhomogeneous term is an exponential function, the particular solution is $y_p = Ae^{2x}$.

$$y_p = Ae^{2x} \quad \rightarrow \quad y_p' = 2Ae^{2x} \quad \rightarrow \quad y_p'' = 4Ae^{2x}$$

Substitute these formulas into equation (2).

$$9(4Ae^{2x}) + Ae^{2x} = e^{2x}$$

$$37Ae^{2x} = e^{2x}$$

Match the coefficients on both sides to get an equation for A.

$$37A = 1$$

Solving it yields

$$A = \frac{1}{37},$$

which means the particular solution is

$$y_p = \frac{1}{37}e^{2x}.$$

Therefore, the general solution to the ODE is

$$\begin{aligned} y(x) &= y_c + y_p \\ &= C_3 \cos\frac{x}{3} + C_4 \sin\frac{x}{3} + \frac{1}{37}e^{2x}, \end{aligned}$$

where C_3 and C_4 are arbitrary constants.